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NUMERICAL SOLUTION OF THE PROBLEM OF THE EFFECT OF A SHOCK PULSE ON AN ICE SHEET

V. D. Zhestkaya and V. M. Kozin

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A mathematical formulation of the problem is given. A method is proposed to determine the initial velocities of points of an ice sheet subjected to a point shock pulse. An example of calculation of ice-sheet deflections is considered.

Key words: *ice sheet, pulse, numerical modeling.*

Investigation of the stress–strain state of an ice sheet subjected to loads of various type makes it possible to solve a number of applied problems arising in navigation on frozen waterways, in operation of engineering facilities in river and offshore areas, and in implementing flood prevention measures during ice drift and freezing periods. In particular, it is of interest to study the behavior of an ice sheet subjected to pulse loading. This problem arises in blasting ice dams and jams and in ice-breaking operations.

At present, analytical solutions of this problem [1, 2] have been found only for relatively simple ice conditions. For problems modeling real ice conditions (arbitrary shorelines, variable water body depth, etc.), the construction of analytical solutions involves great mathematical difficulties, and using numerical methods is therefore more promising.

In the present work, a numerical method combining the finite-element method and the finite-difference method is used to calculate ice-sheet deflections under the action of a point shock pulse, i.e., a suddenly applied force P for a time interval $\Delta\tau$ which is smaller than the period of natural vibrations. According to [1], the ice sheet is represented as a plate and water is considered an ideal incompressible liquid. The water depth is assumed to be constant.

In constructing the mathematical model, we use rectangular coordinates with the x and y axes located in the plane of the ice plate and the z axis directed upward (Fig. 1).

As the basic relations we adopt the differential equation of viscoelastic vibrations of the ice sheet [1]

$$D\Big(1 + \tau_f \frac{\partial}{\partial t}\Big)\nabla^4 w + \rho_w g w + \rho_i h \frac{\partial^2 w}{\partial t^2} + \rho_w \frac{\partial \Phi}{\partial t}\Big|_{z=0} = p(x, y, t), \qquad (1)$$

the Laplace equation for the liquid velocity potential

$$\nabla^2 \Phi = 0, \qquad (2)$$

the nonpenetration condition at the bottom of the water basin

$$\frac{\partial \Phi}{\partial z}\Big|_{z=-H} = 0 \qquad (3)$$

and the condition of equal velocities at the ice–water interface

$$\frac{\partial w}{\partial t} - \frac{\partial \Phi}{\partial z}\Big|_{z=0} = 0. \qquad (4)$$

Komsomol'sk-on-Amur State Technical University, Komsomol'sk-on-Amur, 681013; kks@knastu.ru. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, Vol. 49, No. 2, pp. 152–159, March–April, 2008. Original article submitted August 1, 2006; revision submitted April 12, 2007.

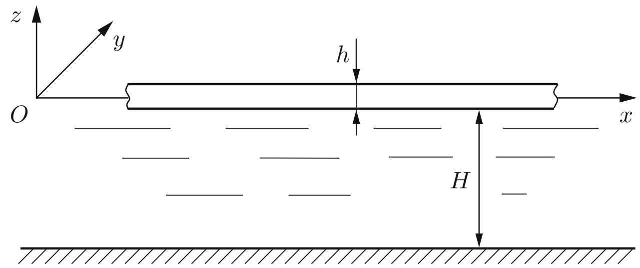

Fig. 1. Diagram of the problem.

Here w is the ice deflection, ρ_w and ρ_i are the densities of water and ice, respectively, g is the acceleration due to gravity, h is the thickness of the ice sheet, Φ is the velocity potential, p is the intensity of the external load, H is the basin depth, τ_f is the strain relaxation time, and D is the flexural rigidity of the plate. In this case,

$$p(x, y, t) = U\delta(x, y)\delta(t),$$

where U is the shock pulse and $\delta(x, y)$ and $\delta(t)$ are the Dirac delta functions.

The solution algorithm is similar to that used in [3, 4] for an ice sheet subjected to a moving load. The numerical solution given in [3, 4] is based on two numerical methods: the finite-element method and the finite-difference method.

The calculation is performed for a relatively large region of liquid with an ice cover which is bounded in the horizontal plane. The area of this region should be sufficient to assume no plate displacements on its boundary Γ and to adopt the fixed-end conditions

$$w\Big|_{(x,y)\in\Gamma} = 0, \qquad \frac{\partial w}{\partial n}\Big|_{(x,y)\in\Gamma} = 0, \tag{5}$$

where n is the normal to the boundary Γ.

On the vertical surface which bounds the liquid under the ice cover, we impose the nonpenetration condition [4]

$$\frac{\partial \Phi}{\partial n}\Big|_{(x,y)\in\Gamma} = 0, \tag{6}$$

where n is the normal perpendicular to the z axis.

Using the method of expanding the displacements in the principal modes, which is known in the theory of small vibrations, we write the expressions for w and Φ in the form

$$w = \sum_{m=1}^{\infty} w_m, \qquad \Phi = \sum_{m=1}^{\infty} \Phi_m,$$

where the functions under the summation sign are linearly independent. Keeping n terms of the series and decomposing the function Φ_m into two factors, we obtain

$$w = \sum_{m=1}^{n} w_m; \tag{7}$$

$$\Phi = \sum_{m=1}^{n} \Phi_m = \sum_{m=1}^{n} \varphi_m(x, y, t)\psi_m(z) = \sum_{m=1}^{n} \varphi_m(x, y, t) \cosh\, (k_m(z + H)). \tag{8}$$

Here $k_m = \text{const}$; the expression for $\psi_m(z)$ was obtained by substituting the relation $\Phi_m = \varphi_m(x, y, t)\psi_m(z)$ into (2), dividing the variables and solving the differential equation with the variable z [4].

It should be noted that, if only n terms of the series are kept, the initial system, which has an infinite number of degrees of freedom, is replaced by a system with a finite number of degrees of freedom n.

With the substitution of (7) and (8) into (1)–(4), Eq. (3) is satisfied. Eliminating φ_m from Eqs. (1), (2), and (4), we obtain the system

$$\sum_{m=1}^{n} \Big(D\Big(1+\tau_f \frac{\partial}{\partial t}\Big)\nabla^4 w_m + \rho_w g w_m + \rho_i h \frac{\partial^2 w_m}{\partial t^2} + \rho_w \frac{\coth k_m H}{k_m} \frac{\partial^2 w_m}{\partial t^2}\Big) = p(x,y,t),$$
$$\frac{\partial}{\partial t}\Big(\frac{\partial^2 w_m}{\partial x^2} + \frac{\partial^2 w_m}{\partial y^2} + k_m^2 w_m\Big) = 0. \tag{9}$$

From (5), we obtain

$$w_m\Big|_{(x,y)\in\Gamma} = 0, \qquad \frac{\partial w_m}{\partial n}\Big|_{(x,y)\in\Gamma} = 0. \tag{10}$$

As shown in [4], if conditions (10) are satisfied on the boundary of the region considered, condition (6) is also satisfied. Thus, we eliminated the potential Φ from the problem and obtained Eqs. (9) and (10) for the function w_m in (7).

Using the finite-element algorithm, we construct a discrete model for the ice plate by setting

$$w_m(x,y,t) = \sum_{i=1}^{n} N_i(x,y) q_{im}(t). \tag{11}$$

Here $N_i(x,y)$ are shape functions, $q_{im}(t)$ are the components of the nodal displacement vector $[q]_m(t)$, and n is the number of nodal displacements and the number of degrees of freedom of the discrete model since, at any point of the discrete model, the displacements are completely determined by the set of nodal displacements. The value of n, which is determined using conditions (10), depends on the type and number of the finite elements constituting the discrete model of the plate.

In view of (11), the expression for the plate deflection becomes

$$w = \sum_{m=1}^{n} w_m = \sum_{m=1}^{n}\sum_{i=1}^{n} N_i(x,y) q_{im}(t) = \sum_{i=1}^{n} N_i(x,y) q_i(t),$$

where $q_i(t) = \sum_{m=1}^{n} q_{im}(t)$ are the components of the total nodal-displacement vector $[q](t)$:

$$[q](t) = \sum_{m=1}^{n} [q]_m(t). \tag{12}$$

The resolving system of equations of the problem is obtained by employing the generalized Bubnov–Galerkin method. As a result, we obtain the system of matrix equations [3, 4]

$$\sum_{m=1}^{n} \Big([M]_m \frac{d^2[q]_m}{dt^2} + [C]\frac{d[q]_m}{dt} + [K][q]_m\Big) = [P](t), \qquad \Big([S] - k_m^2[T]\Big)\frac{d[q]_m}{dt} = 0, \tag{13}$$

where $[P](t)$ is the vector of the external nodal loads. The elements of the matrices $[M]_m$, $[C]$, $[K]$, $[S]$, and $[T]$ in (13) depend on ρ_w, ρ_i, h, H, τ_f, D, k_m, and $N_i(x,y)$.

System (13) is solved using the finite-difference method [3, 4]. After transformations, we obtain the matrix equations

$$\sum_{m=1}^{n} \Big([A]_m[q]_{m,r+1} + [B]_m[q]_{m,r} + [D]_m[q]_{m,r-1}\Big) = (\Delta t)^2 [P]_r,$$
$$\Big([S] - k_m^2[T]\Big)\Big([q]_{m,r+1} - [q]_{m,r-1}\Big) = 0, \qquad r = 0,1,2,\ldots,L, \tag{14}$$

where Δt is the grid step, $[q]_{m,r}$ is the value of the vector $[q]_m$ at the rth node, and $[P]_r$ is the value of the nodal loads vector at $t = r\Delta t$; the matrix coefficients $[A]_m$, $[B]_m$, and $[D]_m$ depend on the parameters of the problem.

The second equation of system (14) can be satisfied by representing the expression for $[q]_{m,r}$ in the form

$$[q]_{m,r} = [X]_m \alpha_{m,r}. \tag{15}$$

Here $[X]_m$ is the eigenvector of the homogeneous system of linear equations with the matrix $[S] - k_m^2[T]$ which correspond to the eigenvalue k_m^2, and $\alpha_{m,r}$ is the unknown coefficient. The eigenvectors $[X]_m$ and the eigenvalues k_m^2 are calculated in the first stage of the calculation using any suitable method (for example, the rotation method).

Substitution of (15) into the first equation of system (14) yields

$$\sum_{m=1}^{n}\Big([A]_m[X]_m\alpha_{m,r+1}+[B]_m[X]_m\alpha_{m,r}+[D]_m[X]_m\alpha_{m,r-1}\Big)=(\Delta t)^2[P]_r,$$
$$r=0,1,2,\ldots,L. \tag{16}$$

Equation (16) should be supplemented by initial conditions. Let, at the initial time $t=0$, the nodal displacement vector $[q]$ be equal to $[f_0]$, and the rate of its change be equal to $[\dot{f}_0]$:

$$[q](0)=[f_0], \qquad \Big(\frac{d[q]}{dt}\Big)\Big|_{t=0}=[\dot{f}_0]. \tag{17}$$

In view of the finite-difference representation of the derivatives, Eqs. (12), (15), and (17) lead to the system of equations

$$\sum_{m=1}^{n}[X]_m\alpha_{m,0}=[f_0], \qquad \sum_{m=1}^{n}[X]_m(\alpha_{m,-1}-\alpha_{m,1})=-2[\dot{f}_0]\Delta t. \tag{18}$$

From Eqs. (18), we obtain $\alpha_{m,0}$ and $\alpha_m=\alpha_{m,-1}-\alpha_{m,1}$. Substituting $\alpha_{m,0}$ and α_m into (16), we obtain the final system of equations for $\alpha_{m,r}$ $(r=1,2,\ldots,L-1)$:

$$\sum_{m=1}^{n}\Big([D]_m+[A]_m\Big)[X]_m\alpha_{m,1}=(\Delta t)^2[P](0)-\sum_{m=1}^{n}[B]_m[X]_m\alpha_{m,0}-\sum_{m=1}^{n}[D]_m[X]_m\alpha_m,$$

$$\sum_{m=1}^{n}[A]_m[X]_m\alpha_{m,r+1}=(\Delta t)^2[P](r\Delta t)-\sum_{m=1}^{n}[B]_m[X]_m\alpha_{m,r}-\sum_{m=1}^{n}[D]_m[X]_m\alpha_{m,r-1}.$$

Knowing $\alpha_{m,r}$, we find the nodal displacements and the plate deflection at node of the time grid.

If the single load on the ice sheet is the shock pulsed load at $t=0$, in the above relations, one should set $[P]=0$ and $[f_0]=0$. The factor initiating the motion of system is the initial velocity $[\dot{f}_0]$ imparted to the ice plate at the time of action of the pulse.

The initial velocities of points of the plate can be determined from the law of conservation of momentum, according to which $K=U$ (K is the momentum acquired by the plate as a result of impact and $U=P\Delta\tau$ is the impact impulse).

To solve the problem in question, we employ the finite-element method. The plate deflection is approximated by the expression

$$w(x,y)=\sum_{i=1}^{n}q_iN_i(x,y),$$

where q_i are the nodal displacements, N_i are shape functions, and n is the number of nodal displacements.

In determining the initial velocities, it is necessary to specify their dependences on the coordinates x and y of points of the plate. We assume that the velocities are proportional to the plate deflections under the static action of the point force P at the point of application of the pulse. Then, the velocity distribution over the plate can be represented as

$$v(x,y)=v_P\,\frac{w(x,y)}{w(x_P,y_P)}, \tag{19}$$

where v_P is the velocity of the plate at the point of impact, $w(x,y)$ is the static deflection of the plate under the action of the force P, $w(x_P,y_P)$ is the static deflection of the plate at the point of application of the force P, and x_P and y_P are the coordinates of the point of application of the force P. We note that in determining the static deflection, we can set $P=1$ since the ratio $w(x,y)/w(x_P,y_P)$ does not depend on the value of P.

The momentum of the plate can be calculated by the formula

$$K=\iint\limits_S\rho_i h v(x,y)\,dx\,dy=\frac{\rho_i h v_P}{w(x_P,y_P)}\iint\limits_S w(x,y)\,dx\,dy,$$

where ρ_i is the density of ice; the integral is taken over the area of the plate S. Denoting the number of finite elements by m, from this formula we obtain

$$K = \frac{\rho_i h v_P}{w(x_P, y_P)} \sum_{i=1}^{m} \iint\limits_{S_i} w_i(x, y)\, dx\, dy, \tag{20}$$

where $w_i(x, y)$ is the deflection of the ith finite element and S_i is the are of the ith finite element.

If the number of nodal displacements of the finite element equals n_i, then

$$w_i(x, y) = \sum_{r=1}^{n_i} q_k^{(i)} N_k^{(i)}(x, y), \tag{21}$$

where $q_k^{(i)}$ are the nodal displacements of the ith finite element and $N_k^{(i)}$ are shape functions.

Substitution of (21) into (20) yields

$$K = \frac{\rho_i h v_P}{w(x_P, y_P)} \sum_{i=1}^{m} \iint\limits_{S_i} \sum_{k=1}^{n_i} q_k^{(i)} N_k^{(i)}(x, y)\, dx\, dy = \frac{\rho_i h v_P}{w(x_P, y_P)} \sum_{i=1}^{m} \sum_{k=1}^{n_i} q_k^{(i)} \iint\limits_{S_i} N_k^{(i)}\, dx\, dy.$$

Introducing the notation $A_k^{(i)} = \iint\limits_{S_i} N_k^{(i)}\, dx\, dy$ and taking into account that $K = U$, we have the following equality for v_P:

$$\frac{\rho_i h v_P}{w(x_P, y_P)} \sum_{i=1}^{m} \sum_{k=1}^{n_i} q_k^{(i)} A_k^{(i)} = U.$$

Having found v_P, from formula (19), we determine the linear velocities of points of the plate, in particular, the velocities of the nodes of the finite-element grid. Thus, we obtain the vector $[\dot{f}_0]$ included in the initial conditions of the problem.

In [3, 4], a method was proposed to calculate the stress–strain state of an ice sheet under the action of a moving load. A comparison of the solutions obtained using this method with available analytical solutions and experimental data obtained in both laboratory and full-scale experiments shows that the numerical solutions are fairly exact [4]. In the present paper, the indicated method is proposed for use in solving the problem of the effect exerted on an ice sheet by a load of a different type, namely, a shock pulse. Because, as noted above, the algorithm for solving this problem is similar to the algorithm for solving the problem of motion of a load on an ice sheet, the latter problem can be regarded as a test for the problem of pulsed load. Reliable results obtained in the case of moving loads suggest that the results obtained for the case of pulsed loads are reliable.

Below, we give results of calculation of plate deflections (obtained in experimental modeling of the response of an ice sheet to pulsed loading) using the method described above. The ice sheet was modeled by a rubber film 1 mm thick which had the shape of a rectangular plate of length $L = 2$ m and width $B = 1.2$ m. The basin depth was 0.02 m. The physical parameters of the problem had the following values: $\rho_w = 1000$ kg/m^3, $\rho_i = 2500$ kg/m^3, $\tau_f = 10$ sec, and $D = 0.14 \cdot 10^{-4}$ nm. The value of the pulse was set equal to $0.98 \cdot 10^{-4}$ N$\cdot$sec, the point of its application was at the center of the plate, and the action of the pulse was directed upward. The time grid step was equal to 0.0625 sec.

The plate was discretized into square finite elements with sides 0.2 m long. The x axis coincided with the symmetry axis of the plate and was parallel to its long edge, and the coordinate origin was on the left edge of the plate. In the calculation, we took into account the symmetry of the plate and considered only its part on one side from the x axis.

The discrete model was constructed using the so-called combined bending rectangular finite element, which had 16 degrees of freedom [5]. At each node of the grid of elements, there were generally four nodal displacements, and at the node lying on the symmetry axis, two nodal displacements. Thus, in this case, in view of the boundary conditions and the symmetry conditions, the total number of nodal displacements was $n = 90$.

Figure 2 shows curves of plate deflections on the x axis at various times from the moment of application of the pulse. The method described above can also be used in the case of several pulses (simultaneously or with a shift in time). We also note that the numerical method chosen for the solution of the problem allows one to take

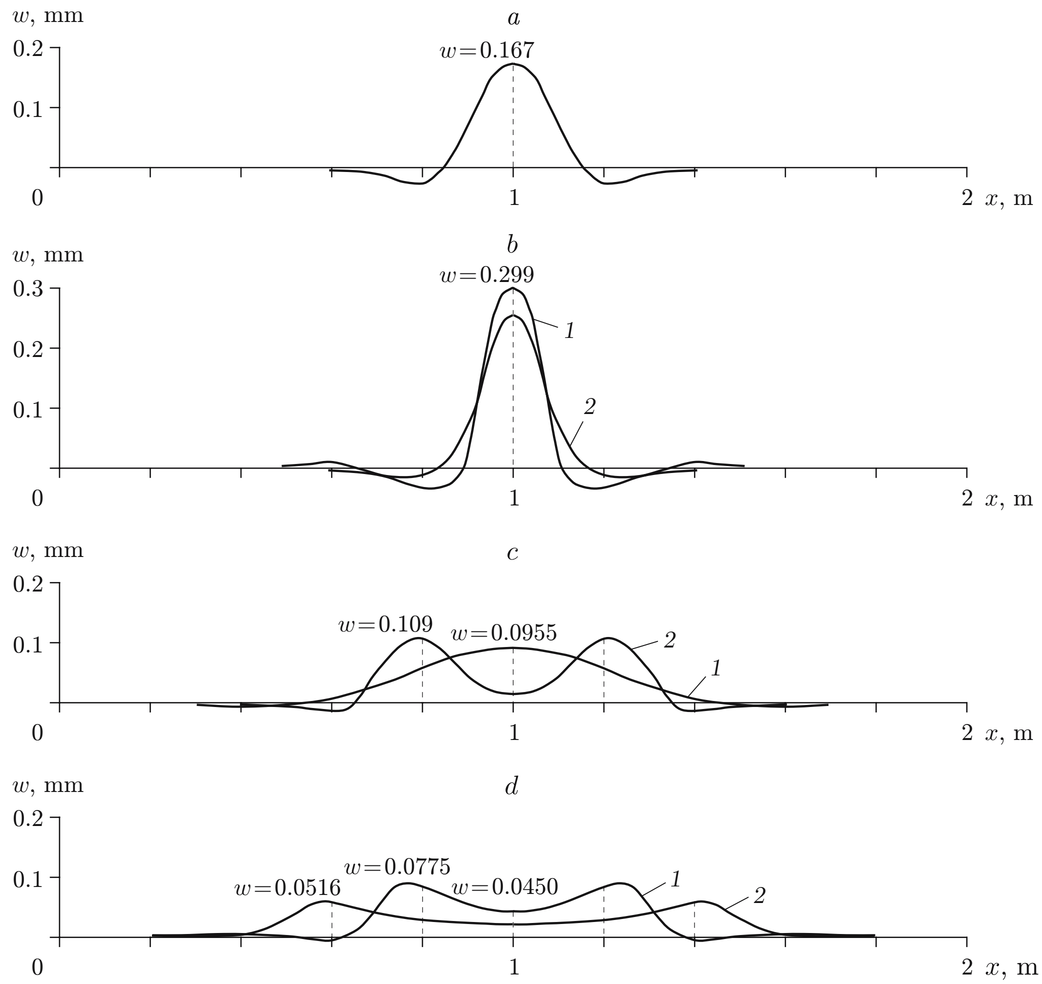

Fig. 2. Ice sheet deflections: (a) $t = 0.0625$; (b) $t = 0.1875$ (1) and 0.25 sec (2); (c) $t = 0.375$ sec (1) and 0.5 sec (2); (d) $t = 0.6875$ (1) and 1.875 sec (2).

into account various features of ice conditions (for example, the basin outline in plan, the presence of cracks, sites of free water, etc.) that are difficult to take into account in the analytical solution of the problem.

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